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A Method of Order Reduction for Structural Dynamics Based on Riccati Iteration

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This paper presents a new method of order reduction based on recent work of a similar nature applicable to system dynamics and control. This method provides a computational procedure for producing a condensed model which exactly preserves the slowest n_1 modes of the total n modes of the original model for almost any set of n_1 degrees of freedom retained in the condensed model. The method can also be used to compute the eigensolutions corresponding to the n_1 slowest modes of the original structural dynamics problem. The accuracy of the condensed model and the speed/accuracy performance of the eigensolver are compared with standard methods for a 90 degree-of-freedom cantilevered plate.

Nomenclature

a = plate boundary length
A = nonsymmetric system matrix, $n \times n$
$\bar{A}$ = transformed system matrix, $n \times n$
A_{ij} = partitions of the A matrix, $n_i \times n_j$
D = plate bending stiffness
D_i = correction matrix, $n_2 \times n_1$
f_i = "fast" eigenvalues, $i = 1,...n_2$
I = identity matrix, $n_1 \times n_1$ or $n_2 \times n_2$
J = diagonal matrix of eigenvalues, $n \times n$
J_i = diagonal partitions of the J matrix, $n_i \times n_i$
k = structure stiffness matrix, $n \times n$
L = Riccati solution matrix, $n_2 \times n_1$
L_i = Riccati iteration matrix, $n_2 \times n_1$
m = structure mass matrix, $n \times n$
M = modal matrix, $n \times n$
M_{ij} = partitions of the M matrix, $n_i \times n_j$
n = number of degrees of freedom
n_1 = number of "slow" eigenvalues
n_2 = number of "fast" eigenvalues
Q = inverse modal matrix, M^{-1}, $n \times n$
Q_{ij} = partitions of the Q matrix, $n_i \times n_j$
R_i = Riccati residual matrix, $n_2 \times n_1$
s_i = "slow" eigenvalues, $i = 1,...n_1$
t = plate thickness
T = linear transformation matrix, $n \times n$
V^i = simultaneous iteration matrix, $n_2 \times n$
V^i_j = partition of the V^i matrix, $n_2 \times n_j$
x = state vector of structural degrees of freedom, $n \times 1$
x_i = partition of the x vector, $n_i \times 1$
y_1 = state vector of the condensed model, $n_1 \times 1$
λ_i = eigenvalues, $i = 1,...n$
μ = system "small parameter"
ρ = plate material density
ϕ_i = eigenvectors, $n \times 1$, $i = 1,...n$
ω_i = natural frequencies, $i = 1,...n$

I. Introduction

SEVERAL methods have been developed previously for order reduction, or condensation, of structural dynamics problems.[1-3] These methods are all approximate in the sense that the normal modes (natural frequencies and mode shapes) calculated from a reduced model differ from the corresponding normal modes of the original system. A poor choice of degrees of freedom (dof) retained in the reduced model may result in very inaccurate modes, while a good choice may produce quite accurate modes, depending on the specific structure being analyzed. But even the best condensation with methods developed previously produces modes that are inaccurate to some degree.

This paper will present a new method of condensation based on the Riccati iteration algorithm recently employed in control system studies.[4,5] This method produces a condensed model which exactly preserves the first n_1 modes of the total n modes of the original model, regardless of which n_1 dof are retained in the condensed model. The price paid for the accuracy of this method is a computational effort generally greater than that required for other condensations. However, if only a moderately accurate condensation is required, the level of accuracy can be decreased by adjusting the convergence parameter in the algorithm and thereby reducing computation time.

The Riccati iteration algorithm can also be used as an eigensolver in structural dynamics problems where only a few low frequencies and mode shapes are desired. The numerical algorithm has a unique solution point, and is globally convergent to this solution. The rate of convergence of the algorithm is determined by the eigenvalue spectrum, and if one has estimates of the range of the slow spectrum, the system matrix can be shifted to increase the convergence rate.

II. Order Reduction Based on the L-Transformation

The homogeneous matrix equation for an undamped structure with n dof is

$$m\ddot{x} + kx = 0$$

where x is the displacement vector consisting of all n dof, and m and k are mass and stiffness matrices, respectively. The corresponding generalized eigenvalue problem is

$$k\phi_i = \omega_i^2 m\phi_i, \quad i = 1,...n$$

If the mass matrix is nonsingular, an equivalent form for the structural vibration problem is

$$\ddot{x} + Ax = 0 \tag{1}$$

where $A = m^{-1}k$. Then the complete eigensolution of Eq. (1) satisfies the equations

$$AM = MJ, \quad A = MJM^{-1} \tag{2}$$

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where M is the modal matrix whose columns are eigenvectors and J is a diagonal matrix of eigenvalues $\lambda_i=\omega_i^2$ arranged from smallest to largest magnitude. In order to describe the condensation, partition matrices M and $Q=M^{-1}$ as

$$M=\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},\quad Q=\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

where M_{11} and Q_{11} are $n_1\times n_1$, etc. A reduced $n_1\times n_1$ matrix which exactly preserves the first n_1 modes of the original model can be written by definition from Eq. (2) as

$$\tilde{A}_{11}=M_{11}J_1M_{11}^{-1} \tag{3}$$

It has been shown[4,5] that the appropriate state y_1 for the condensation

$$\ddot{y}_1+\tilde{A}_{11}y_1=0 \tag{4}$$

is related to the original state

$$x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ as } y_1=M_{11}[Q_{11}x_1+Q_{12}x_2]$$

and the transformation from y_1 back to x is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} I \\ M_{21}M_{11}^{-1} \end{bmatrix}y_1$$

The condensation matrix $\tilde{A}_{11}$, Eq. (3), is not obtained by computing M_{11} and J_1 directly, but rather by computing the L-transformation matrix described subsequently. The obvious drawback to computing a condensed model with Eq. (3) is that n_1 modes (eigensolutions) of the original large problem are required. But the primary purpose for order reduction is to eliminate the necessity for solving such large eigenvalue problems. So Eq. (3) represents the desired product of condensation but not the method for computing that condensation.

A canonical transformation of variables taken from singular perturbations literature[6] can be employed to reduce A to block-triangular form. Specifically, define the transformation matrix T as

$$T=\begin{bmatrix} I & 0 \\ L & I \end{bmatrix} \tag{5}$$

where L is a real $n_2\times n_1$ matrix and the I are identity matrices of order n_1 and n_2, with $n_1+n_2=n$. If we partition the A matrix conformably with Eq. (5) as

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{6}$$

then the transformed matrix

$$\tilde{A}=\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}=TAT^{-1}$$

$$=\begin{bmatrix} A_{11}-A_{12}L & A_{12} \\ LA_{11}-A_{22}L-LA_{12}L+A_{21} & A_{22}+LA_{12} \end{bmatrix} \tag{7}$$

will be upper block-triangular provided the L matrix satisfies the nonsymmetric algebraic Riccati equation (NARE)

$$0=LA_{11}-A_{22}L-LA_{12}L+A_{21} \tag{8}$$

Then, since $\tilde{A}$ is upper block-triangular, the eigenvalues of A become the eigenvalues of $\tilde{A}_{11}$ and $\tilde{A}_{22}$, i.e.,

$$\lambda(A)=\{\lambda_1,\ \lambda_2,\ldots\lambda_n\}=\lambda(\tilde{A}_{11})\cup\lambda(\tilde{A}_{22})$$

In general there will be one solution L of NARE corresponding to each possible partitioning of the eigenvalues of A between $\lambda(\tilde{A}_{11})$ and $\lambda(\tilde{A}_{22})$. However, the particular solution L which is of greatest interest in the condensation problem is that which separates the $\lambda(A)$ into slow eigenvalues

$$\lambda(A_{11}-A_{12}L)=\{\lambda_1,\ldots,\lambda_{n_1}\}=\{s_1,\ldots,s_{n_1}\} \tag{9a}$$

and fast eigenvalues

$$\lambda(A_{22}+LA_{12})=\{\lambda_{n_1+1},\ldots,\lambda_n\}=\{f_1,\ldots,f_{n_2}\} \tag{9b}$$

It has been shown[4,5] that the matrix L which satisfies both NARE and the time-scale separation property, Eqs. (9a) and (9b), is unique and may be expressed as

$$L=-M_{21}M_{11}^{-1}=Q_{22}^{-1}Q_{21} \tag{10}$$

provided M_{11} is nonsingular.

III. Computation of the L Matrix Through Riccati Iteration

One method of computing the L matrix is through the Riccati iteration

$$L_{i+1}=(A_{22}+L_iA_{12})^{-1}(L_iA_{11}+A_{21}) \tag{11}$$

which can be initialized with $L_0=0$. This iteration method for computing L has very attractive numerical features. If we define the system small parameter as the ratio of eigenvalues

$$\mu=s_{n_1}/f_1 \tag{12}$$

where $\mu<1$ provided $s_{n_1}\neq f_1$, then[4,5] iteration Eq. (11) has asymptotic rate of convergence μ. That is, if we define the Riccati residual

$$R_i=L_iA_{11}-A_{22}L_i-L_iA_{12}L_i+A_{21} \tag{13}$$

then

$$\lim_{i\to\infty}\frac{\|R_{i+1}\|}{\|R_i\|}=\mu \tag{14}$$

in any convenient matrix norm $\|\cdot\|$. This rate of convergence is clearly illustrated in the succeeding examples.

It has also recently been shown[7] that the Riccati iteration is directly related to the globally convergent subspace iteration method.[8] This relationship is derived in the Appendix.

The Riccati iteration Eq. (11) can be rewritten in a form yielding greater numerical accuracy. At each iteration of Eq. (11) we can compute a correction matrix D_i and update the L_i matrix as

$$L_{i+1}=L_i+D_i$$

The matrix D_i is then formed from the Riccati residual R_i as

$$L_{i+1}=L_i+D_i=(A_{22}+L_iA_{12})^{-1}(L_iA_{11}+A_{21})$$

$$D_i=(A_{22}+L_iA_{12})^{-1}(L_iA_{11}-A_{22}L_i-L_iA_{12}L_i+A_{21})$$

$$=(A_{22}+L_iA_{12})^{-1}R_i$$

To summarize, the Riccati iteration to obtain the slow mode condensation

$$\tilde{A}_{11}=A_{11}-A_{12}L$$

consists of the following steps.

1) Choose n_1 and convergence tolerance $\epsilon>0$, and set $L_0=0$, $i=0$.

2) Evaluate residual $R_i=L_iA_{11}-A_{22}L_i-L_iA_{12}L_i+A_{21}$ and go to step 5 if $\|R_i\|<\epsilon$.

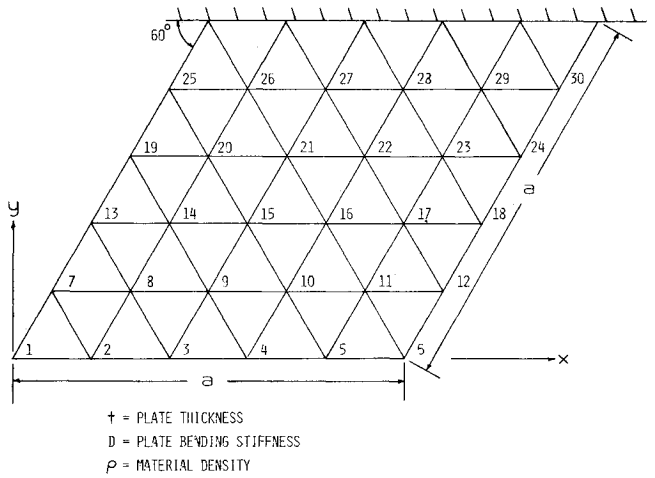

Fig. 1 90 dof cantilevered parallelogram plate for the numerical example.

Table 1 Low frequency eigenvalues for the 90 dof parallelogram plate

i	λ_i	$\mu(n_1=i)$	μ'
1	15.9	0.174	—
2	91.7	0.135	0.061
3	679.9	0.934	0.873
4	728.1	0.406	0.251
5	1791.3	0.665	0.496
6	2692.0	0.456	0.295
7	5897.3	0.947	0.898
8	6229.7	0.744	0.591
9	8376.7	0.776	0.633
10	10800.9	—	—

3) Solve $(A_{22}+L_iA_{12})D_i=R_i$ for D_i by LU decomposition and let $L_{i+1}=L_i+D_i$.

4) Increase index i by one and go to step 2.

5) Compute the condensation $\tilde{A}_{11}=A_{11}-A_{12}L$.

If one's purpose is not simply to obtain a condensation (for simulation or other purposes) but rather to compute the slow mode eigenvalues $\{s_i\}$ and slow mode eigenvectors

$$\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}$$

then proceed with the following steps.

6) Compute the n_1 eigenvalues $\{s_i\}$ and n_1 eigenvectors M_{11} of the condensation $\tilde{A}_{11}$.

7) Obtain the lower partition of the n_1 slow eigenvectors as $M_{21}=-LM_{11}$.

IV. Improvements to the Riccati Iteration

As noted earlier, the rate of convergence of the Riccati iteration is determined by the ratio of fast and slow eigenvalues, Eq. (12). If one has estimates $\hat{s}_1$, $\hat{s}_{n_1}$ of the smallest and largest slow eigenvalues, then one can shift the A matrix as

$$A-\lambda_s I=A'=\begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix} \tag{15}$$

where $\lambda_s=(\hat{s}_1+\hat{s}_{n_1})/2$, and use A' rather than A in the Riccati iteration. The resulting asymptotic rate of convergence will be

$$\mu'=\frac{\max\{\,|s_1-\lambda_s|,\ |s_{n_1}-\lambda_s|\,\}}{|f_1-\lambda_s|}<\mu$$

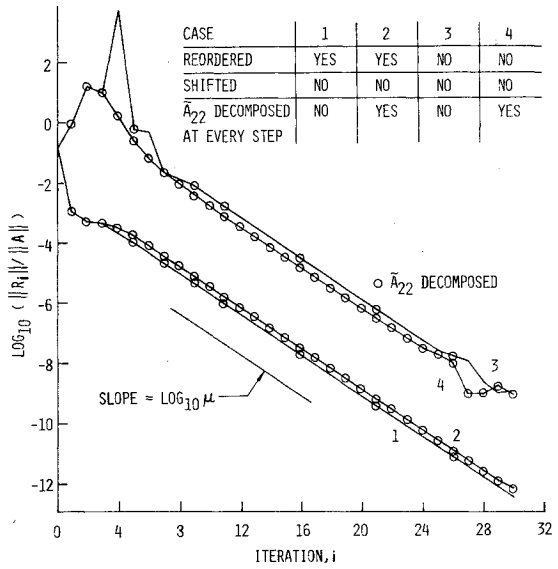

Fig. 2 Convergence of the Riccati iteration with $n_1=6$ for the 90 dof example.

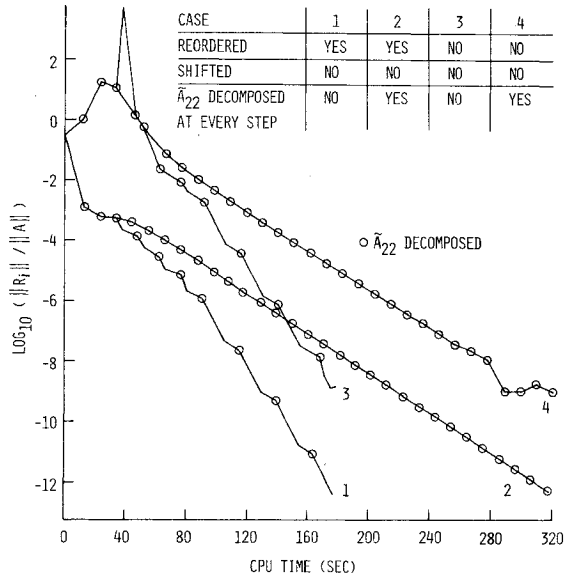

Fig. 3 Computation times for the Riccati iteration with $n_1=6$ for the 90 dof example.

which will be particularly beneficial for the case where slow eigenvalues $\{s_1,\ldots,s_{n_1}\}$ are closely spaced. However, the shift does not change the L matrix and the condensation can still be computed as in step 5.

The computation time for the Riccati iteration can also be decreased by updating the decomposition of the large $n_2\times n_2$ matrix $(A_{22}+L_iA_{12})$ only when D_i is large. As the iteration proceeds, the successive D_i become small and one need not update the LU decomposition of $(A_{22}+L_iA_{12})$ at each iteration. The savings in computation time yielded by these modifications to the algorithm are demonstrated in the numerical example.

V. Numerical Example

The structure used for this example is the cantilevered, parallelogram, uniform, thin plate in bending shown in Fig. 1. This structure was selected because it is known[9] to have closely spaced third and fourth natural frequencies, which might pose a challenge to the condensation and eigenvalue methods being examined. The plate was modeled with equilateral triangle finite elements, as shown in Fig. 1, and

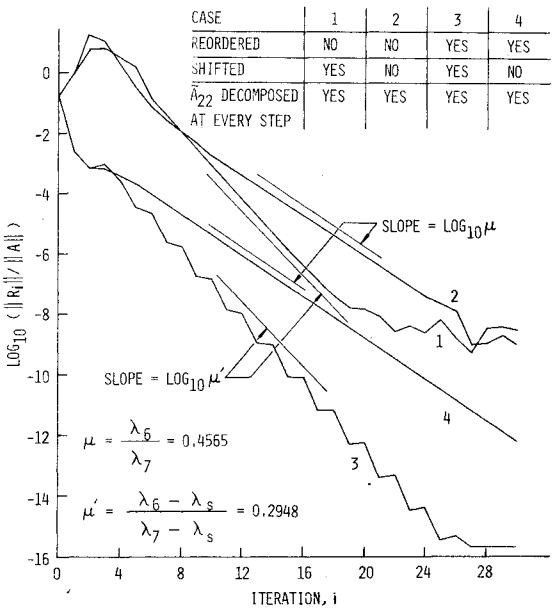

Fig. 4 Convergence of the shifted Riccati iteration with $n_1 = 6$ for the 90 dof example.

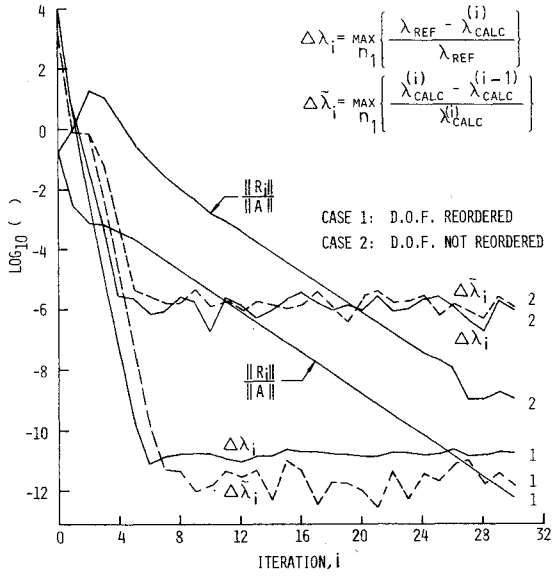

Fig. 5 Eigenvalue convergence for the Riccati iteration; reordered, unshifted, $\tilde{A}_{22}$ decomposed at every step.

each of the 30 unrestrained nodes was given freedom to tranlate out of the xy plane and to rotate about the x and y axes in the plane, for a total of 90 dof. Condensed NASTRAN[10] with CTRIA2 bending elements was used to generate the stiffness and consistent mass matrices. The numerical eigenvalues discussed in this section are dimensionless, being equal to the dimensional squared natural frequencies divided by $D/\rho t a^4$.

Table 1 lists the low frequency eigenvalues for the parallelogram plate and shows the values of μ and μ' corresponding to different choices of n_1. The number n_1 was chosen as 6 for this study so $\mu = 0.456$. Note that it would be very unfortunate to choose n_1 as 7, yielding $\mu = 0.947$ and correspondingly slow convergence.

Figures 2-4 illustrate the performance of the Riccati algorithm for the structure of Fig. 1. All computations were performed in double precision (approximately 14 decimal digits) on an IBM 370/158 computer using the CMS timesharing system. The convergence of the algorithm is measured by monitoring the size of the Riccati equation

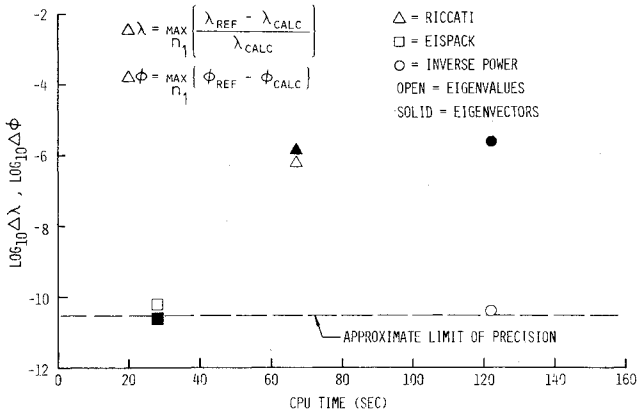

Fig. 6 Time/accuracy comparison with other methods, computing 6 eigensolutions.

Table 2 Relative eigenvalue error for Guyan condensation

		Reordered		Nonreordered	
i	Actual λ_i	Guyan λ_i	Error, %	Guyan λ_i	Error, %
1	15.95	16.13	1.1	17.11	7.3
2	91.67	95.43	4.1	216.2	135.9
3	679.9	731.9	7.7	2179.4	>200
4	728.1	1141.4	56.7	78,366.	>200
5	1791.3	2445.9	36.5	192,048.	>200
6	2692.0	9768.1	>200	not found	—

residual, normalized as $\log_{10}(\|R_i\|/\|A\|)$. The matrix norm used in these computations was the Euclidean norm, i.e.,

$$\|B\| = \left[\sum_i \sum_j b_{ij}^2\right]^{1/2}$$

An important question that arises in the construction of structural condensations is which dof should be placed in x_1. If one can choose n_1 dof which strongly represent the n_1 slow modes so that $\|M_{11}\|$ is large, then $\|L\|$ will be small and the algorithm will converge more rapidly. However, for all cases tested, the algorithm converged for all sets of dof placed in x_1. The reordered and nonreordered cases in Fig. 2 correspond, respectively, to x_1 including either translation dof at nodes 1, 4, 6, 13, 18, and 24, or all dof at nodes 1 and 2. Also, Fig. 2 shows that one need decompose $(A_{22}+L_iA_{12})$ only a few times in the initiation of the algorithm to achieve the analytical convergence rate.

The decomposition of $(A_{22}+L_iA_{12})$ is one of the most computationally expensive steps in the algorithm, and computation times are significantly decreased by decomposing $(A_{22}+L_iA_{12})$ only infrequently, as shown in Fig. 3.

The shifted Riccati iteration employing A' rather than A, cf. Eq. (15), converges with a higher asymptotic rate, as shown in Fig. 4. In both shifted and unshifted cases the analytical convergence rates μ and μ' accurately predict the performance of the algorithm.

The accuracy of the eigensolutions of the condensation $\tilde{A}_{11}$ increases rapidly as the Riccati residual $\|R_i\|$ decreases, as illustrated in Fig. 5. Note that the eigenvalue error decreases more rapidly than the Riccati residual, and the former may be used as a criterion to terminate the iteration. The "reference" eigensolutions in Figs. 5 and 6 were computed by subroutine EIGRF of the International Mathematics and Statistics Library (IMSL). At convergence the eigensolutions of $\tilde{A}_{11}$ were accurate to six or more decimal digits for the nonreordered case and ten or more decimal digits for the reordered case. The corresponding accuracy of eigensolutions for Guyan condensation[3] is listed in Table 2. As shown, the accuracy of Guyan condensation is strongly dependent on the

particular dof's retained, and is considerably less than the accuracy of the Riccati condensation.

To evaluate the relative efficiency of the Riccati iteration as an eigensolver compared to standard methods, the first six modal frequencies and shapes were also computed for the 90 dof plate using 1) the inverse power method with shifts[11] and 2) the Cholesky factorization/Sturm sequencing algorithm coded in EISPACK.[12] The resulting accuracy/computation time performances are summarized in Fig. 6. In almost all calculations reported here, the symmetry, bandedness, and sparseness of structural matrices were not exploited to improve computational efficiency, with the exception that the EISPACK routines used were specialized for symmetric m and k matrices. Therefore, these performance figures are meaningful only as comparative rather than absolute measures.

VI. Summary

The recently proposed Riccati iteration algorithm has been applied to small-sized structural dynamics problems to produce accurate low-order condensations. The numerical algorithm is attractive in its simplicity and global convergence to the desired solution. However, good rate of convergence is dependent upon choosing the number n_1 of slow modes corresponding to a gap in the eigenvalue spectrum so that $\lambda_{n1}/\lambda_{n1+1}$ is small.

The Riccati iteration may also be used as an eigensolver to compute slow eigenvalues and eigenvectors but is shown to be somewhat slower than standard computational methods in this limited test. There are several possibilities for effecting refinements such that it may be made competitive. The algorithm can be adjusted easily to yield eigensolutions containing only a few decimal places of accuracy and requiring correspondingly shorter computation times.

More details of this research are reported in Ref. 13. Additional research remains to be done to take full advantage of symmetry, bandedness, sparseness, and mass matrix diagonality in the Riccati iteration.

Appendix

There is a direct relationship between the Riccati iteration Eq. (11) and the subspace iteration method. The subspace iteration method includes a power iteration step, a projection step, and an eigensolution step. The power iteration can be represented as

$$V^{i+1} = V^i A \tag{A1}$$

Provided none of the slow eigenvectors

$$\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}$$

is orthogonal to the rows of V^0, the $n_2 \times n$ matrix V^i converges to a basis of the subspace spanned by the n_2 fast left eigenvectors $[Q_{21}Q_{22}]$.[14] To demonstrate the equivalence of Eqs. (11) and (A1), partition V^i compatibly with A as

$$[V_1^{i+1}V_2^{i+1}] = [V_1^i V_2^i]\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

so

$$V_1^{i+1} = V_1^i A_{11} + V_2^i A_{21}$$

$$V_2^{i+1} = V_1^i A_{12} + V_2^i A_{22}$$

Assuming V_2^i and V_2^{i+1} are nonsingular, it follows that

$$(V_2^i)^{-1} V_1^{i+1} = (V_2^i)^{-1} V_1^i A_{11} + A_{21}$$

$$(V_2^i)^{-1} = ((V_2^i)^{-1} V_1^i A_{12} + A_{22})(V_2^{i+1})^{-1}$$

Thus, we have

$$(V_2^{i+1})^{-1} V_1^{i+1}$$

$$= (A_{22} + (V_2^i)^{-1} V_1^i A_{12})^{-1} ((V_2^i)^{-1} V_1^i A_{11} + A_{21})$$

which for $L_i = (V_2^i)^{-1} V_1^i$ shows the equivalence of Eqs. (11) and (A1) and implies global convergence of Eq. (11). Note that L_i converges to $Q_{22}^{-1}Q_{21}$, whereas V^i converges to $[Q_{21}Q_{22}]$.

Acknowledgment

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